Macroeconomic Implications of the Ramsey Model:  
Consumer Optimisation and Endogenous Growth

マクロ経済学におけるラムゼイ・モデルの示唆：  
消費者最適化行動と内生成長

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Abstract: Since the criticisms Lucas made in ‘Economic Policy Evaluation: A Critique (1976)’ became widely accepted in the 1970s and 1980s, ‘micro foundations’ have been recognised as essential for macroeconomic models, and consequently, the dynamic stochastic general equilibrium (DSGE) model developed remarkably in the 1990s. It is a common idea that research on the DSGE models was originated from the real business cycle (RBC) model created by Kydland and Prescott (1982). However, preceding the development of the RBC model, the Ramsey-Cass-Coopmans model (simply ‘the Ramsey model’ or ‘the optimal growth model’) introduced micro foundations and the optimal behaviour of households as an economic growth model, and lead to consumption theories such as the permanent income hypothesis, investment theories such as Tobin’s q, and the RBC model. This paper demonstrates the process of setting the basic Ramsey model to observe its features and, by observing empirical examples of cross-country convergence, its speed is also examined. Further, necessary conditions for generating endogenous growth are discussed along with limitations of the Ramsey models being assessed and with other endogenous growth models being introduced.

Keywords: Ramsey model, Solow model, endogenous growth, cross-country convergence

要約: 1976年のいわゆる『ルーカス批判』は1980年代から90年代にかけて幅広く認知され、マクロ経済モデルに「ミクロ的基礎付け」は必須のものと認識されるようになった。その結果、動学的確率的一般均衡（DSGE）モデルは1990年代にめざましい発展を遂げる。DSGEモデルは、1982年にKydlandとPrescottによって確立された実物的景気循環（RBC）モデルから始まるとは周知の通りだが、恒常所得仮説などの消費理論、トービンのqなどの投資理論、そしてRBCモデルの先駆けになったのは、ミクロ的基礎付けや家計の最適化行動を取り入れたラムゼイ・モデル（最適成長モデル）である。本稿では、第二章において基本的なラムゼイ・モデルを構築するプロセスを示しながらその特徴を明らかにし、第三章では国家横断的な収束スピードを論ずる。また第四章では、ラムゼイ・モデルの限界と他の内生成長モデルを紹介しつつ、内生成長をもたらすために必要な条件を論ずる。

キーワード: ラムゼイ・モデル、ソロー・モデル、内生成長、国家横断的収束
1. Introduction

With numerous analyses of neoclassical growth models, there is little doubt that the Solow-Swan (simply, the Solow) model played significant roles in the neoclassical growth studies. Yet, as analysed in Barro and Sala-i-Martin (2004), it can be said that one of the shortcomings of the Solow-Swan model is an exogenous and constant saving rate. If we are not allowed to consider optimal behaviour of consumers, we cannot discuss how the economy reacts to changes such as interest rates, tax rates, and so forth. In the Solow-Swan model, the optimal behaviour of firms is allowed to be considered. However, it does not change any basic results mainly because the total amount of investment in the economy is still given by the saving rate which remains exogenous.

Thus, households optimising their utility and firms interacting on competitive markets need to be taken into account so as to analyse the process of economic growth more accurately. In the Ramsey model, created by Ramsey (1928) and followed by Cass (1965) and Koopman (1965), infinitely-lived households consume and save in order to maximise their utility subject to an intertemporal budget constraint. This consumer optimisation is one of key elements which enables the Ramsey model to more accurately analyse how households should distribute between consumption and savings and accumulate capitals by investing their savings and is ideal in an economy.

Also, the Ramsey model can evaluate the efficiency of process of economic growth whereas the Solow model cannot analyse what process of economic growth or capital accumulation is most appropriate in an economy.

2. Set-up of the Ramsey Model

Following Barro and Sala-i-Martin (2004), by constructing the Ramsey model, this section implies how it introduces the optimal behaviour of consumers and becomes more ‘forward-looking’ than previous growth models represented by the Solow model.

2.1 Households

Households generally behave as follows. They:

1) provide labour service and receive wages,
2) receive interest income on assets,
3) purchase goods for consumption,
4) save by accumulating assets.

Here we assume identical households, which have the same preference parameters
and the same wage rate, start from the same assets per person, and have the same rate of population growth.

To simplify, we also assume the family size grows at the exogenous and constant rate \( n \). Normalising the number of adults at time 0 to unity, the family size at time \( t \) is given by,

\[
L(t) = e^{nt}.
\]

(1)

Let \( C(t) \) be total consumption at time \( t \), consumption per adult person \( c(t) \) is showed as,

\[
c(t) = \frac{C(t)}{L(t)}.
\]

(2)

Here each household tries to maximise overall utility \( U \), which is given by,

\[
U = \int_0^\infty u[c(t)]e^{nt}e^{-\rho t}dt,
\]

(3)

where the rate of time preference \( \rho > 0 \), means that utilities are valued less when received later. Also, we assume that \( u(c) \) is increasing in \( c \) and concave, that is,

\[
\begin{align*}
&u'(c) > 0 \\
&u''(c) < 0
\end{align*}
\]

and that \( u(c) \) satisfies Inada conditions such as;

\[
\begin{align*}
&u'(c) \to \infty \quad \text{as} \quad c \to 0 \\
&u'(c) \to 0 \quad \text{as} \quad c \to \infty
\end{align*}
\]

Next, let \( r(t) \) and \( w(t) \) be the interest rate as given and the wage rate per adult paid per unit of labour services, respectively. Then the total income received by the aggregate of households will be the sum of labour income, \( w(t) \cdot L(t) \), and asset income, \( r(t) \cdot (\text{Assets}) \).

This thus leads to the following equation:

\[
\frac{d(\text{Assets})}{dt} = r \cdot (\text{Assets}) + wL - C,
\]

(4)

which means that households use the income that they do not consume to accumulate more assets.

Let \( a \) be per capita assets, then we have

\[
a = \left( \frac{1}{L} \right) \left[ \frac{d(\text{Assets})}{dt} \right] - na.
\]

(5)

Substituting equation (4) into (5), we get the budget constraint in per capita terms given by

\[
a = w + ra - c - na.
\]

(6)
2.2 First-order conditions

We begin with the present-value Hamiltonian given by
\[ J = u[c(t)] \cdot e^{-(\rho-n)t} + \nu(t) \cdot \{w(t) + [r(t) - n] \cdot a(t) - c(t)\}, \]
where \( \nu(t) \) is the present value shadow price income, which represents the value of an increment of income received at time \( t \) in units of utilities at time 0.

The first-order conditions for a maximum of \( U \) are showed as
\[
\frac{\partial J}{\partial c} = 0 \Rightarrow \nu = u'(c)e^{-(\rho-n)t} \\
\nu = -\frac{\partial J}{\partial a} \Rightarrow \nu = -(r - n) \cdot \nu
\]
Also, the transversality condition is given by
\[
\lim_{t \to \infty} [\nu(t) \cdot a(t)] = 0.
\]

2.3 The Euler equation

The Euler equation itself was originally derived by Swiss mathematician Euler in the seventeenth century and first derived by Ramsey and Keynes in an economic context. This is a condition of efficient dynamic distribution of consumption; thus, it is the core of consumer optimisation.

From the first-order conditions as mentioned above, we obtain the Euler equation, which shows the basic condition for choosing consumption over time as below.
\[
r = \rho - \frac{du'(c)}{u(c)} = \rho - \left[ u''(c) \cdot \frac{c'}{c} \right] \cdot \left( \frac{c'}{c} \right)
\]
It is common to assume the constant intertemporal elasticity of substitution (CIES) utility function showed as below in such a case.
\[
u(c) = \frac{c^{1-\theta}}{1-\theta} - 1
\]
Substituting equation (10) into (9), the optimality condition (9) simplifies to
\[
c'/c = (1/\theta) \cdot (r - \rho)
\]
where \((1/\theta)\) is constant. \((1/\theta)\) represents how much the growth rate of consumption reacts the difference between the real interest rate and the rate of time preference, and therefore, it is called the CIES as mentioned above. Equation (11) implies that, if \( r > \rho \), the consumers are given an incentive to save more at the present time for more future consumption.
2.4 The transversality condition

The transversality condition (8),

\[ \lim_{t \to \infty} [v(t) \cdot a(t)] = 0, \]

implies the value of the household’s per capita assets, that is, the \( a(t) \) times the shadow price \( v(t) \) has to approach 0 as time approaches infinity. In other words, utility would increase if the assets were used to raise consumption at some time in finite time.

Let us suppose unrealistically that everyone knows that the world will end at some known date \( T > 0 \). If the shadow value price at time \( T \) \( v(T) > 0 \), then households will optimise their utility by selling all the assets to raise consumption when the world ends, that is, \( a(T) = 0 \).

Integrating equation (7) with respect to time, we obtain the following.

\[ v(t) = v(0) \cdot \exp \left\{ - \int_0^t [r(v) - n]dv \right\} \]

Then substituting it into equation (8), we get the reformed transversality condition as

\[ \lim_{t \to \infty} \left\{ a(t) \cdot \exp \left\{ - \int_0^t [r(v) - n]dv \right\} \right\} = 0. \quad (12) \]

2.5 Firms

Firms generally behave as follows. They:

1) produce goods,
2) pay wages for labour input,
3) make rental payment for capital input.

Here, each firm has the production function

\[ Y(t) = F[K(t), L(t), T(t)], \quad (13) \]

where \( Y \) is the flow of output, \( K \) is capital input in units of commodities, \( L \) is labour input in person-hours per year, and \( T(t) \) is the level of technology assumed to grow constant rate \( x \geq 0 \). Thus, \( T(t) = e^{xt} \).

Considering the labour-augmenting form, equation (13) can be rewritten as

\[ Y(t) = F[K(t), L(t) \cdot T(t)] . \quad (14) \]

If we define ‘effective labour’ as \( \hat{L} \equiv L \cdot T(t) \), the production function can be restated as

\[ Y = F[K, \hat{L}]. \quad (15) \]

Also, we deal with \( Y \) and \( K \) per unit of effective labour given by
\[ \hat{Y} \equiv Y / \hat{L}, \]
\[ \hat{k} \equiv K / \hat{L}, \]

and then, the production function can be rewritten as
\[ \hat{y} = f(\hat{k}), \]
where \( f(0) = 0. \)

Next, when the representative firm’s flow of profit at any point in time is given by
\[ \pi = F(K, L) - (r + \delta) \cdot K - wL, \]
this equation can be written as
\[ \pi = \hat{L} \cdot [f(\hat{k}) - (r + \delta) \cdot \hat{k} - we^{-\mu}]. \]

Given \( r \) and \( w \), a competitive firm maximises profit for given \( \hat{L} \) by setting
\[ f'(\hat{k}) = r + \delta \]
Also, in a full-market equilibrium, \( w \) is equal to the marginal product of labour and the following condition can be showed when the value of \( k \) satisfies equation (19).
\[ [f(\hat{k}) - \hat{k} \cdot f'(\hat{k})]e^{-\mu} = w \]
This implies that profit is equal to zero for any value of \( \hat{L} \).

### 2.6 Equilibrium
From the household’s flow budget constraint in equation (6) which determines \( \hat{a}, a = k, \)
\[ \hat{k} = ke^{-\mu}, \]
and the conditions for \( r \) and \( w \) showed in equations (19) and (20) respectively, we obtain the following equation,
\[ \dot{\hat{k}} = f(\hat{k}) - \hat{c} - (x + n + \delta) \cdot \hat{k}, \]
where \( \hat{c} = C / \hat{L} = ce^{-\mu} \) and \( \hat{k}(0) \) is given. This equation determines the evolution of \( \hat{k} \) but does not mention the determination of \( \hat{c} \). The Solow-Swan model assumes a constant saving rate as \( \hat{c} = (1 - s) \cdot f(\hat{k}) \). However, we already know that \( c \) grows as equation (11) shows for household optimisation.

Thus, using the conditions \( r = f'(\hat{k}) - \delta \) and \( \hat{c} = ce^{-\mu} \), we obtain
\[ \frac{\dot{\hat{c}}}{\hat{c}} = \frac{\dot{\hat{c}}}{c} - x = \frac{1}{\theta} \left[ f'(\hat{k}) - \delta - \mu - \theta \xi \right]. \]
Considering the initial condition, \( \hat{k}(0) \), and the transversality condition, the combination of equations (21) and (22) forms a system of two differential equations in \( \hat{c} \) and \( \hat{k} \), and the system determines their time path.

Also, substitute \( a = k \) and \( \hat{k} = ke^{-\mu} \) into equation (12) and we get the transversality condition in terms of \( k \) given by
\[
\lim_{t \to \infty} \left\{ \dot{k} \cdot \exp \left(-\int_{0}^{\infty} \left[f'(\hat{k}) - \delta - x - n \right] d\nu \right) \right\} = 0
\] 

(23)

When the initial consumption ratio is below \( \dot{c}(0) \), the initial saving rate is too high to remain on the saddle path in `the phased diagram of the Ramsey model.' The fact that households are oversaving means that the transversality condition is violated, and accordingly, paths which have initial consumption ratio below \( \dot{c}(0) \) are not equilibria. In sum, the role of the transversality condition is important to determine the unique equilibrium.

2.7 The steady state

Deriving equations (21), (22), and (23), we are now ready to consider the steady state. First, let \( \left( \gamma_k \right) \) denote the steady state growth rate of \( \dot{k} \) and \( \left( \gamma_c \right) \) the steady state growth rate of \( \dot{c} \). In the steady state, we get from equation (21)

\[
\dot{c} = f'(\hat{k}) - (x + n + \delta) \cdot \hat{k} - \hat{k} \cdot (\gamma_c)^{\ast}.
\] 

(24)

Differentiating the above with respect to time,

\[
\dot{\gamma} = \dot{k} \cdot \left[f'(\hat{k}) - (x + n + \delta) \cdot \hat{k} - \hat{k} \cdot (\gamma_c)^{\ast} \right]
\] 

(25)

Considering the transversality condition (23), \( \left( \gamma_k \right) \) and \( \left( \gamma_c \right)^{\ast} \) must have the same sign. If \( \left( \gamma_k \right) > 0, \hat{k} \to \infty \) and \( f'(\hat{k}) \to 0 \). Then equation (22) implies \( \left( \gamma_c \right)^{\ast} < 0 \), which is inconsistent with the fact that \( \left( \gamma_k \right) \) and \( \left( \gamma_c \right)^{\ast} \) have the same sign. Similarly, if \( \left( \gamma_k \right) < 0, \hat{k} \to 0 \) and \( f'(\hat{k}) \to \infty \). Then equation (22) implies \( \left( \gamma_c \right)^{\ast} > 0 \), which is also inconsistent with the fact that \( \left( \gamma_k \right) \) and \( \left( \gamma_c \right)^{\ast} \) have the same sign. Thus, the only possibility is \( \left( \gamma_k \right)^{\ast} = \left( \gamma_c \right)^{\ast} = 0 \).

The result \( \left( \gamma_k \right)^{\ast} = 0 \) leads \( \left( \gamma_c \right)^{\ast} = 0 \). To sum up, these results imply the following:

1) \( \hat{k}, \dot{c}, \) and \( \dot{y} \) are constant in the steady state.
2) \( k, c, \) and \( y \) grow in the steady state at the rate \( x \).
3) \( K, C, \) and \( Y \) grow in the steady state at the rate \( x + n \).

3. Convergence

A key issue in economic growth studies has been whether poor countries or regions tend to grow faster than rich ones. In the Solow model, the countries with smaller initial per capita income show a higher growth rate. In other words, poorer countries always catch up with richer countries, and finally, all the countries realise the same level of per capita income. Does this ‘unconditional convergence’ reflect the reality?
In this section, we first examine the speed of convergence in the Ramsey model. Following the steps of setting up the model demonstrated in Section Two, the key features of convergence in the Ramsey model for closed economies are showed in what follows.

We assume that the production function is Cobb-Douglas, that is,

\[ f(\dot{k}) = A \cdot \dot{k}^a, \]

where \(0 < \alpha < 1\).

By using a log linearization of equation (21) and (22), the solution for \( \log[\dot{y}(t)] \) in the log-linearized approximation to the model with a Cob-Douglas technology is given by

\[ \log[\dot{y}(t)] = e^{-\beta t} \cdot \log[\dot{y}(0)] + (1 - e^{-\beta t}) \cdot \log(y^*), \]

where \(\beta > 0\). This parameter decides the speed of adjustment to the steady state and is given by

\[ 2\beta = \left\{ \zeta^2 + 4 \cdot \left( \frac{1 - \alpha}{\theta} \right) \cdot (\rho + \delta + \theta \kappa) \cdot \left[ \frac{\rho + \delta + \theta \kappa}{\alpha} - (n + x + \delta) \right] \right\}^{\frac{1}{2}} - \zeta \]

where \(\zeta = \rho - n - (1 - \theta) \cdot x > 0\). Thus the average growth rate of \(y\) over the interval between an initial time \(0\) and any future time \(T \geq 0\) is showed as

\[ \frac{1}{T} \cdot \log[y(T)/y(0)] = x + \frac{(1 - e^{-\beta t})}{T} \cdot \log[y^* / \dot{y}(0)]. \]

This equation implies that the higher \(\beta\), the wider the gap between \(\log(y^*)\) and \(\log[\dot{y}(0)]\), that is, there is a faster convergence to the steady state.

According to Barro and Sala-i-Martin (1992), diminishing returns to capital are a significant element for convergence in the neoclassical model.

In fact, their empirical results prove the existence of convergence in the sense that economies tend to grow faster when they are further below the steady-state level. This phenomenon can be clearly observed in their results on U. S. states from 1840 to 1988 and in a sample of 98 countries from 1960 to 1985, although some conditions are required. For example, taking different given saving rates or population growth rates into account, this ‘conditional convergence’ is observed not only in their studies but also in Mankiw, Romer, and Weil (1992).
4. Conditions for Generating Endogenous Growth

Table: “Growth Rate of Per Capita GDP by Major Region
(annual average compounded growth rate)”

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Japan</td>
<td>0.88</td>
<td>4.05</td>
<td>1.88</td>
</tr>
<tr>
<td>Western Europe</td>
<td>1.56</td>
<td>2.45</td>
<td>1.84</td>
</tr>
<tr>
<td>Latin America</td>
<td>1.43</td>
<td>2.58</td>
<td>0.91</td>
</tr>
<tr>
<td>Asia (excl. Japan)</td>
<td>-0.10</td>
<td>2.91</td>
<td>3.55</td>
</tr>
<tr>
<td>Africa</td>
<td>0.92</td>
<td>2.00</td>
<td>0.90</td>
</tr>
</tbody>
</table>


(See the website of Maddison, A., http://www.ggdc.net/maddison/)

In the Solow model, the growth rate of per capita income decreases over time due to diminishing returns to capital. However, the table shows that the growth rates of per capita income have not constantly decreased since 1913 in the same regions, and that there are huge gaps in growth rates among countries and regions at the same time.

Also, in the Ramsey model as well as the Solow model, the steady-state per capita grows at the rate of technical progress, $\chi$, which we assume exogenous. The Ramsey model is therefore helpful to study transitional dynamics. However, it does not clearly refer to the sources of long-term growth of per capita income.

While these ‘exogenous growth models’, in which the growth rate is determined with exogenous population growth rate or technical progress rate, ‘endogenous growth models’ have been developed to more clearly explain these empirical studies. One way to explain endogenous growth is to eliminate diminishing returns to capital. In this section, the AK model is briefly introduced as a simple example in which we assume constant returns to capital.

4.1 Brief setup of the AK model

First, infinite-lived households maximise utility given by

$$U = \int_0^\infty e^{-(\rho-n)t} \left[ \frac{c^{1-\theta} - 1}{(1-\theta)} \right] dt,$$

subject to the constraint

$$\dot{a} = (r-n) \cdot a + w - c.$$
Then, conditions for optimisation are again given by
\[
\dot{c}/c = (1/\theta) \cdot (r - \rho).
\]

Also, the transversality condition is again given by
\[
\lim_{t \to \infty} \left\{ a(t) \cdot \exp \left[ - \int_0^t [r(v) - \nu] dv \right] \right\} = 0.
\]

Second, when we consider the behaviour of firms, we need to assume the linear production function
\[
y = f(k) = Ak,
\]
where \( A > 0, f'' = 0 \), which means that marginal product of capital is not diminishing and that this production function differs from the neoclassical one which assumes diminishing returns to capital.

Following the process of equilibrium and transitional dynamics, we obtain one of the key equations in the AK model given by
\[
\dot{k}/k = \dot{c}/c = (1-\theta) \cdot (A-\delta - \rho),
\]
where changes in \( A, \rho, \) and \( \theta \) affect the levels and growth rates of \( c \) and \( k \).

### 4.2 Determinants of the growth rate

As showed above, in the AK model, the long-run growth rate depends on the parameters such as \( A, \rho, \) and \( \theta \), which determine the per capita growth rate. For instance, lower values of \( \rho \) and \( \theta \) raise the willingness to save and achieve a higher per capita growth rate. Another example is a technological change, namely, an improvement in parameter \( A \) also raising the growth rate.

In contrast, in the Ramsey model, the long-run per capita growth rate is fixed at the value of \( x \), the exogenous rate of technological change. The different results between these two models are due to the presence or absence of diminishing returns to capital.

Therefore, it can be said that technological progress is a key element for generating endogenous growth.

In addition, another interpretation of the AK model is that capital should be viewed broadly to include both physical and human components. Some models, see for instance Arrow (1962), Uzawa (1965), Romer (1986), and Lucas (1988), focus on human capital affected by technological change through experience on productivity and knowledge spillovers.
5. Conclusion

The arguments considered in this paper infer that one way to generate endogenous growth is to eliminate diminishing returns to capital. Further, taking into account human capitals or technological progress is also effective for further analyses of endogenous growth. Finally, we derived the above while bearing in mind that although the Ramsey model implies that long-run average growth rate is zero, economies are known in reality to grow for hundreds of years.

References


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